

Lesson

4-7

Scale-Change Images of Trigonometric Functions

Vocabulary

sine wave

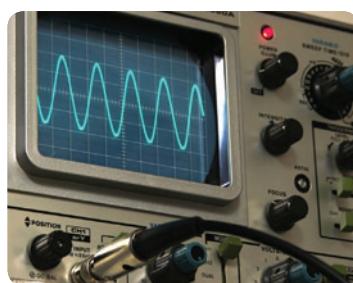
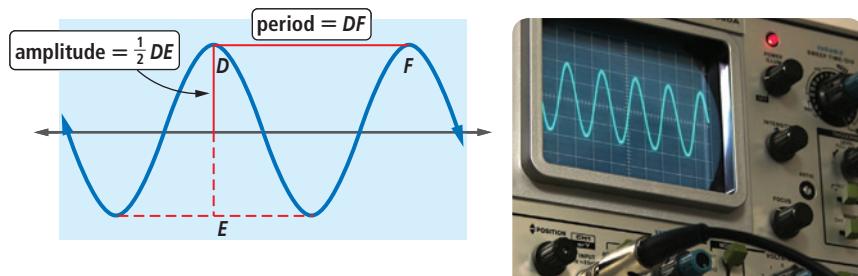
amplitude

frequency

► **BIG IDEA** The Graph Scale-Change Theorem can be applied to obtain the equation, amplitude, and period of scale-change images of the graph of a parent trigonometric function.

Sine Waves

A pure tone, such as that produced by a tuning fork, travels in a *sine wave*. A **sine wave** is the image of the graph of the sine or cosine function under a composite of translations and scale changes. The pitch of the tone is related to the period of the wave; the longer the period, the lower the pitch. The intensity of the tone is related to the *amplitude* of the wave. The **amplitude** of a sine wave is half the distance between its maximum and minimum values.

**STOP QY1**

Stretching a sound sine wave vertically changes the intensity of the tone. Stretching a sound wave horizontally changes its pitch. Recall from Lesson 3-5 that a scale change is a mapping $S: (x, y) \rightarrow (ax, by)$ centered at the origin with $a \neq 0$ and $b \neq 0$. Under this scale change, an equation for the image of $y = f(x)$ is $\frac{y}{b} = f\left(\frac{x}{a}\right)$.

Example 1 shows how a scale change affects both the amplitude and period of a sine wave.

Mental Math

The expression $3r - 8$ is in the form $ax - h$. Rewrite it in the form $\frac{x - \frac{h}{a}}{\frac{1}{a}}$.

An oscilloscope is used to record changes in the voltage of an electric current.

QY1

The range of the sine function is $\{y \mid -1 \leq y \leq 1\}$, so the amplitude of the graph of $y = \sin x$ is _____.

Example 1

Consider the function with equation $y = 6 \cos\left(\frac{x}{3}\right)$.

- Explain how this function is related to its parent function, the cosine function.
- Identify its amplitude and its period.

(continued on next page)

Solution

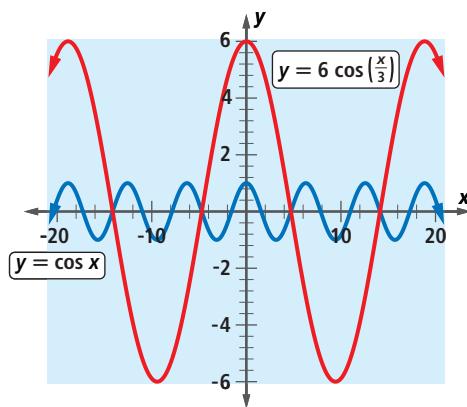
- a. Divide each side of the given equation by 6. This rewrites the function rule in a form that can be analyzed using the Graph Scale-Change Theorem.

$$\frac{y}{6} = \cos\left(\frac{x}{3}\right)$$

In the equation $y = \cos x$, y has been replaced by $\frac{y}{6}$ and x by $\frac{x}{3}$. The graph of the parent function is stretched vertically by a factor of 6 and horizontally by a factor of 3.

- b. The vertical stretch means that the maximum and minimum values of the parent function are multiplied by 6. Hence, the given function has amplitude $\frac{1}{2}(6 - -6) = 6$. The horizontal stretch means that the period 2π of the parent function is also stretched by a factor of 3. So the given function has a period of $3(2\pi) = 6\pi$.

Check Graphs of parts of $y = \cos x$ and $y = 6 \cos\left(\frac{x}{3}\right)$ are shown at the right. Only a little more than two cycles of $y = 6 \cos\left(\frac{x}{3}\right)$ are shown, but from this you can see that the amplitude and period found above were correct.



If the graphs of $y = \cos x$ and $y = 6 \cos\left(\frac{x}{3}\right)$ in Example 1 represented sound waves, the sound of $y = 6 \cos\left(\frac{x}{3}\right)$ would be 6 times as loud and have a lower pitch than that of $y = \cos x$.

In general, the functions defined by $\frac{y}{b} = \sin\left(\frac{x}{a}\right)$ and $\frac{y}{b} = \cos\left(\frac{x}{a}\right)$ are equivalent to the functions defined by

$$y = b \sin\left(\frac{x}{a}\right) \text{ and } y = b \cos\left(\frac{x}{a}\right),$$

where $a \neq 0$ and $b \neq 0$, and their graphs are images of the graphs of the parent functions

$$y = \sin x \quad \text{and} \quad y = \cos x$$

under the scale change that maps (x, y) to (ax, by) . The theorem below indicates the relationship of the constants a and b to the properties of the sine waves.

Theorem (Properties of Sine Waves)

The graphs of the functions defined by $y = b \sin\left(\frac{x}{a}\right)$ and $y = b \cos\left(\frac{x}{a}\right)$ have amplitude $= |b|$ and period $= 2\pi|a|$.

STOP QY2

There is a corresponding theorem for the graph of the function with equation $\frac{y}{b} = \tan\left(\frac{x}{a}\right)$. However, the parent tangent function does not have an amplitude and the period of the parent tangent function is π , so the period of $\frac{y}{b} = \tan\left(\frac{x}{a}\right)$ is $\pi|a|$.

► QY2

In the theorem, why is absolute value used in the calculation of both amplitude and period?

Example 2

The graph at the right shows an image of the graph of $y = \sin x$ under a scale change. Find an equation for the image.

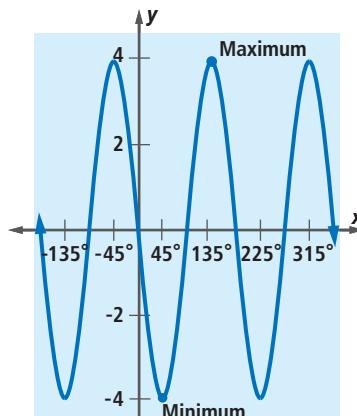
Solution An equation for the image is of the form $y = b \sin(\frac{x}{a})$. From the graph, there is a minimum at $(45^\circ, -4)$ and a maximum at $(135^\circ, 4)$. The difference between the maximum and minimum values of y is 8, so the amplitude is 4. The graph shows a cycle from 0° to 180° , so the period is 180° . Therefore, $|b| = 4$ and $360^\circ |a| = 180^\circ$. Thus, $b = 4$ or -4 and $a = \frac{1}{2}$ or $-\frac{1}{2}$. Consider the four possibilities.

$$y = 4 \sin(2x)$$

$$y = -4 \sin(2x)$$

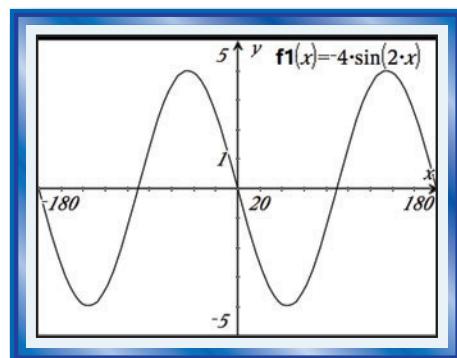
$$y = 4 \sin(-2x)$$

$$y = -4 \sin(-2x)$$



Notice that the graph pictured must be a reflection image of the graph of the parent sine function since, for example, starting at zero, the graph decreases as you move towards 45° . One equation that will produce the graph is $y = -4 \sin(2x)$.

Check Use a graphing utility to check that your equation produces the given graph.



QY3

The Frequency of a Sine Wave

Notice that the graph of $y = \cos x$ in Example 1 completes 3 cycles for every one completed by the graph of $y = 6 \cos(\frac{x}{3})$. We say that $y = \cos x$ has three times the *frequency* of $y = 6 \cos(\frac{x}{3})$. In general, the **frequency** of a periodic function is the reciprocal of the period, and represents the number of cycles the curve completes per unit of the independent variable. Thus, the frequency of the cosine function is $\frac{1}{2\pi}$, and the frequency of the function $y = 6 \cos(\frac{x}{3})$ is $\frac{1}{6\pi}$.

When a sine wave represents sound, doubling the frequency results in a pitch one octave higher. So the graph of $y = 6 \cos(\frac{x}{3})$ represents a sound with pitch between one and two octaves lower than the pitch represented by $y = \cos x$ and with 6 times the intensity. It is common in these situations to view the x -axis as representing time. In sound waves, the y -axis represents pressure, typically measured in newtons (abbreviated N) per square meter, $\frac{N}{m^2}$.

► QY3

Which of the other choices in the solution will produce the graph?

Example 3

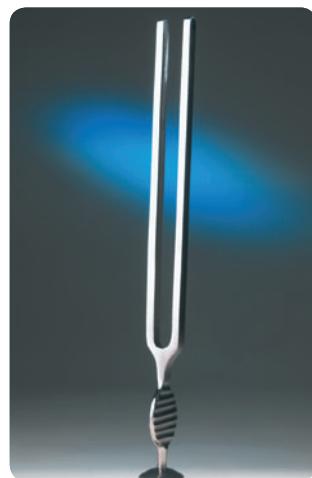
A tuning fork vibrates with a frequency of 512 cycles per second. The intensity of the tone is the result of a vibration whose maximum pressure is $22 \frac{N}{m^2}$. Find an equation to model the sound wave produced by the tuning fork.

Solution The equation has the form $y = b \sin\left(\frac{x}{a}\right)$ where x is the time in seconds after the tuning fork is struck. The frequency is the reciprocal of the period, so

$$512 = \frac{1}{2\pi|a|}.$$

Solve for a to get $a = \pm \frac{1}{1024\pi}$.

The maximum pressure of the air gives the amplitude $b = 22$. Choosing the positive value of a , the equation is $y = 22 \sin(1024\pi x)$.



Tuning In Tuning forks are most commonly used to tune musical instruments to the note “A.”

Knowing the number of cycles per unit of independent variable can help you solve trigonometric equations.

Example 4

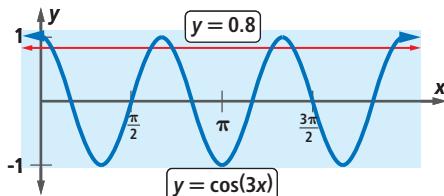
Without using technology, determine how many solutions each equation below has on the interval $0 \leq x \leq 2\pi$. Confirm your answer with a graph.

a. $\cos(3x) = 0.8$

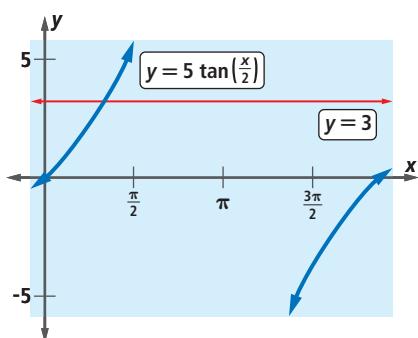
b. $5 \tan\left(\frac{x}{2}\right) = 3$

Solution

- a. The parent cosine function has two solutions in the interval $0 \leq x \leq 2\pi$ because this domain represents one cycle. $y = \cos(3x)$ is the image of $y = \cos x$ under a horizontal shrink of magnitude $\frac{1}{3}$. That means that each cycle of $y = \cos(3x)$ is one-third as long as a cycle of $y = \cos x$, so there will be three cycles on the interval $0 \leq x \leq 2\pi$ and six solutions. A graph confirms that there are six points of intersection of $y = \cos(3x)$ and $y = 0.8$ on $0 \leq x \leq 2\pi$.



- b. The graph of $y = 5 \tan\left(\frac{x}{2}\right)$ is the image of the graph of $y = \tan x$ under a horizontal stretch by a factor of 2. So each cycle is twice as long, and there are half as many points of intersection on the interval $0 \leq x \leq 2\pi$. Therefore, there is only one solution. The graph at the right confirms a single solution.



Questions

COVERING THE IDEAS

- Consider the function with equation $y = \frac{1}{5} \sin x$.
 - True or False** The graph of this function is a sine wave.
 - What is its period?
 - What is its amplitude?
 - Sketch graphs of $y = \frac{1}{5} \sin x$ and $y = \sin x$ on the same set of axes for $-\pi \leq x \leq 2\pi$.
 - Describe how the two graphs in Part d are related.

In 2 and 3, an equation for a sine wave is given.

- Find its amplitude.
- Find its period.
- $y = 3 \cos x$
- $\frac{y}{4} = \cos\left(\frac{x}{3}\right)$

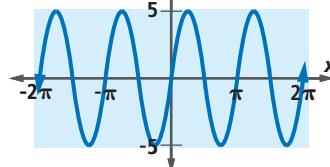
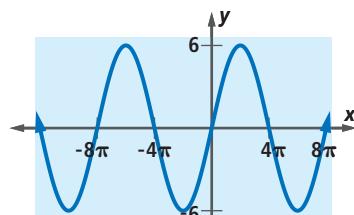
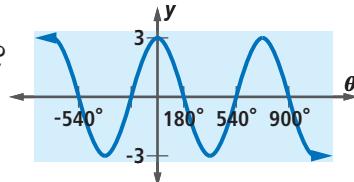
- Multiple Choice** Which equation could yield the graph at the right?

- A $y = 3 \sin(2\theta)$ B $y = 3 \cos(2\theta)$
 C $y = 3 \cos\left(\frac{\theta}{2}\right)$ D $y = 3 \sin\left(\frac{\theta}{2}\right)$

- a. Find an equation of the image of the graph of $y = \sin x$ under the transformation $(x, y) \rightarrow (5x, y)$.
 b. Find the amplitude and period of the image.
- a. Give the period and amplitude of $y = \frac{1}{5} \sin(3\theta)$.
 b. Check using a graphing utility.
- The graph at the right is an image of the graph of $y = \sin x$ under a scale change. Find an equation for this curve.
- Refer to the graph sketched at the right.
 a. Identify the amplitude, period, and frequency.
 b. **Fill in the Blanks** If this graph represents a sound wave, then that sound is ? times as loud and has ? times the frequency of the parent sound wave with equation $y = \sin x$.
- Suppose one tone has a frequency of 330 cycles per second, and a second has a frequency of 660 cycles per second.
 a. Which has the higher pitch?
 b. How many octaves higher is that pitch?
- How many solutions does $\sin(5x) = 0.65$ have for $0 \leq x \leq 2\pi$?

- Sketch one cycle of $6y = \cos\left(\frac{x}{4}\right)$, and label the zeros of the function.
- Multiple Choice** A sound wave whose parent is the graph of $y = \sin x$ has 3 times the frequency and 7 times the amplitude of the parent. What is a possible equation for this sound wave?

A $y = 7 \sin(3x)$ B $y = 7 \sin\left(\frac{1}{3}x\right)$
 C $y = 3 \sin\left(\frac{1}{7}x\right)$ D $y = \frac{1}{3} \sin\left(\frac{1}{7}x\right)$

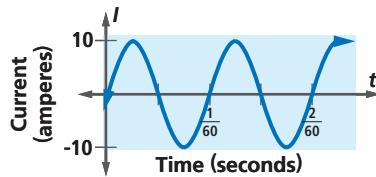


Most modern full-size pianos have 88 keys spanning $7\frac{1}{4}$ octaves.

13. Consider a tuning fork vibrating at 440 cycles per second and displacing air molecules by a maximum of $32 \frac{\text{N}}{\text{m}^2}$. Give a possible equation for the sound wave that is produced.

APPLYING THE MATHEMATICS

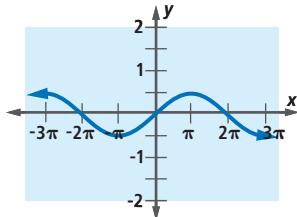
14. Residential electricity is called AC for “alternating current,” because the direction of current flow alternates through a circuit. The current (measured in amperes) is a sine function of time. The graph at the right models an AC situation.
- Write an equation for current I as a function of time t .
 - Find the current produced at 0.04 seconds.
15. Which of the functions f , g , and h , defined by $f(x) = \tan x$, $g(x) = \tan(3x)$, and $h(x) = 3 \sin(2x)$, have the same period?



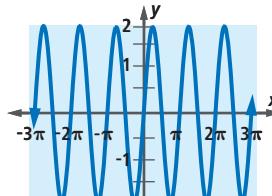
In 16–18, match each equation with its graph below.

16. $\frac{y}{2} = \sin\left(\frac{x}{2}\right)$ 17. $2y = \sin\left(\frac{x}{2}\right)$ 18. $\frac{y}{2} = \sin(2x)$

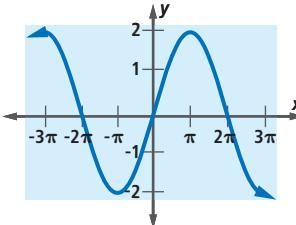
A



B



C



REVIEW

19. Given that $\tan 0.675 \approx 0.8$, find three other values of θ with $\tan \theta \approx 0.8$. (Lesson 4-6)
20. Find the exact value of $\tan(-120^\circ)$. (Lesson 4-4)
21. Give the radian equivalent to each. (Lesson 4-1)
- -720°
 - 225°
 - 315°
22. State the Graph-Translation Theorem. (Lesson 3-2)
23. a. Graph $f(x) = x^3$ and its image under the translation $T: (x, y) \rightarrow (x + 3, y - 1)$.
b. Find an equation for the image. (Lesson 3-2)

EXPLORATION

24. *Pitch* and *loudness* are common words for the frequency and amplitude of sound. Light waves are also modeled with trigonometric functions.
- What properties of light do the frequency and amplitude of light waves represent?
 - Name some other characteristics that sound waves and light waves share.

Q& ANSWERS

- 1
- Both are distances and cannot be negative.
- $y = 4 \sin(-2x)$